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Phase space quantisation of interacting vortices in two dimensions

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Abstract. We have studied the quantisation of point vortices in two dimensions by several alternative symmetrisation rules, and have found that only one rule yields a self-consistent quantum model. The symmetrisation rule is motivated by the attempt to construct an operator that represents the singular classical vorticity, and yields a quantum model which exhibits the effect of a finite vortex core at short distances. A generalisation of the Onsager–Feynman circulation theorem is obtained which reflects the idea that a vortex cannot be localised to within an accuracy greater than that allowed by the Heisenberg principle, and the combination of these two principles yields a vortex core size of the order of an inter-particle spacing. The model is also used to study the energy spectrum of a pair of interacting vortices. For vortices of equal circulation a discrete spectrum is obtained, reflecting the oscillator symmetry, while the energy of a pair with opposite circulation varies continuously with separation, reflecting translational symmetry in a fluid of infinite extent. In both cases the interaction energy is asymptotically logarithmic at large separations, but varies quadratically at small separations, reflecting the effect of a finite core size due to zero-point motion. The theory predicts that two vortices with opposite circulation can annihilate at short distances with an interaction energy that vanishes as the square of their separation.

1. Introduction

Classical models of vortices with quantised circulation (Onsager 1949) have been useful in understanding many properties of bulk ^4He, and have recently been used to study the dissipation in thin ^4He films with thickness of the order of several inter-particle spacings (Ambegaokar *et al* 1978). From a classical viewpoint the fluid particles outside the vortex core ($r \geqslant a$) undergo circular motion with a speed equal to

$$v = \kappa/2\pi r, \tag{1}$$

where r is the distance from the vortex and κ is the circulation ($\kappa = h/m$ for a quantised vortex (Onsager 1949)). Two point vortices in two dimensions (2D) with circulations κ_1 and κ_2 will have an energy given by (Onsager 1949)

$$H(r_{12}) = -\rho(\kappa_1\kappa_2/2\pi)\log r_{12}, \tag{2a}$$

where ρ is the 2D fluid density and r_{12} is the distance between the vortices. In the spirit of equation (1), equation (2a) is valid only when the vortex cores are distinct, $r_{12} \geqslant a$, where the core size a is not predicted by the classical theory (we assume that real vortices have finite cores). The parameter a is generally assumed to be at least of the order of an inter-particle spacing $a \geqslant (\rho/m)^{-1/2}$, since this is the smallest length scale available, even in the quantum many-body theory of ^{4}He. One of the goals of the present work has been to show how quantisation of the classical hydrodynamic theory leads to a vortex core size of the order of an inter-particle spacing, even if we quantise a classical point vortex. There are two essential points behind this idea. First, the classical hydrodynamics of point vortices in 2D can be written in Hamiltonian form, and the rectangular position coordinates (X_i, Y_i) are (to within a constant factor) canonically conjugate variables (q_i, p_i) (Onsager 1949). If we quantise the system, X_i and Y_i will become non-commuting operators which obey an uncertainty principle (Mittag and Stephen 1968). This means that the point vortex cannot be localised to within an accuracy greater than that allowed by the uncertainty principle, so that a finite vortex core will emerge. The smallest core size will be determined by the zero-point motion. These ideas were first suggested in the context of another problem (McCauley 1974), but were not developed at that time. The attempt to quantify these ideas leads us to the second point.

Consider the velocity field due to a classical point vortex located at the point $\boldsymbol{r}'$ rather than at the origin. Equation (1) is replaced by $v = \kappa(2\pi)^{-1}|\boldsymbol{r}-\boldsymbol{r}'|^{-1}$, but more importantly $\boldsymbol{v}(\boldsymbol{r})$ satisfies the classical equation

$$\nabla \times \boldsymbol{v}(\boldsymbol{r}) = \kappa \hat{\boldsymbol{z}} \delta(\boldsymbol{r}-\boldsymbol{r}'), \tag{3}$$

where $\hat{\boldsymbol{z}}$ is a unit vector perpendicular to the plane of the fluid and $\kappa\delta(\boldsymbol{r}-\boldsymbol{r}') = w(\boldsymbol{r}, \boldsymbol{r}')$ is the vorticity or vortex density. This vortex density describes a point vortex whose location is precisely known. Suppose, however, that $\boldsymbol{r}'$ is a classical random variable. For simplicity we can consider the case where $\boldsymbol{r}'$ is defined by a Gaussian distribution with width (RMS fluctuation) σ. Then the *average* vorticity will be Gaussian, and the uncertainty in position gives rise to an effective finite core size $a \sim \sigma$. The details are given in the Appendix. Our interest at this point is directed toward the question of how a similar effect might arise quantum mechanically, where the vortex location $\boldsymbol{r}' = (X', Y')$ is not a classical random variable, but is defined by two non-commuting operators.

Our task would seem to be simple at first sight. We can take equation (3) seriously, even in the quantum case where (X', Y') are replaced by non-commuting operators, if we can find an operator that represents the 2D delta function $\delta(\boldsymbol{r}-\boldsymbol{r}')$. This can indeed be accomplished, but the resulting success is not trouble-free: the replacement of $\delta(\boldsymbol{r}-\boldsymbol{r}')$ by an operator is not unique. This non-uniqueness seems to be due to our starting point, which is an attempt to quantise hydrodynamics rather than starting with the currently insolvable quantum many-body theory of a ^{4}He film. If we can face and successfully resolve the problem of non-uniqueness, it is possible that the resulting quantum hydrodynamical model may be useful as a semiclassical approximation, and may serve as a guide for further work from the many-body viewpoint, especially since the problem of non-commuting vortex position variables has not been considered in that context. At the very least our resulting model will be conceptually superior to the classical point vortex, since it emerges with a statistically defined vortex core.

We will show in § 3 how the non-uniqueness can be resolved and how a sensible model can be constructed. First, we will review briefly the classical Hamiltonian dynamics of 2D point vortices, since this theory is the basis for our quantum model. Then, after introducing the model in § 3, we will use it to predict the energy spectrum and dynamics of an interacting pair of point vortices, first for a vortex pair with equal circulation, and then for a pair with opposite circulation. The predictions for a pair with equal circulation differ quantitatively from those of an earlier theory (Mittag and

Stephen 1968), but not qualitatively. The results for a pair with opposite circulation are entirely new. In § 6 we discuss the problem of non-uniqueness further, since it is also the source of the quantitative discrepancy between our results and those of Mittag and Stephen (1968). We show that an extension of the Mittag–Stephen method to the case of oppositely circulating vortices yields classical point vortex behaviour as $r \to 0$, whereas our model reflects the effect of a finite core (the fluid kinetic energy vanishes as the vortex separation goes to zero).

2. Hamiltonian dynamics of 2D vortices

The work of Kirchhoff (1883) first showed how Hamiltonian mechanics for point vortices in 2D could be obtained from classical hydrodynamics. Consider an unbounded fluid and let κ_i and (X_i, Y_i) denote respectively the circulation and rectangular position coordinates of the ith vortex. The velocity field of the fluid at a point $\boldsymbol{r}$ is defined by $\nabla \cdot \boldsymbol{v}(\boldsymbol{r}) = 0$ (incompressible fluid) and by

$$\nabla \times \boldsymbol{v}(\boldsymbol{r}) = \hat{\boldsymbol{z}} \sum_i \kappa_i \delta(\boldsymbol{r} - \boldsymbol{r}_i), \tag{4}$$

where $\boldsymbol{r}_i = (X_i, Y_i)$ and the sum is over all vortices in the fluid. If one integrates $(\rho/2)\boldsymbol{v}^2(\boldsymbol{r})$ (the kinetic energy density) over the area of the fluid, then, aside from logarithmically infinite self-energy terms, the fluid kinetic energy is (Mittag and Stephen 1968)

$$H = -\frac{\rho}{2\pi} \sum_{i<j} \kappa_i \kappa_j \log r_{ij}, \tag{5}$$

where $r_{ij} = |\boldsymbol{r}_i - \boldsymbol{r}_j|$ is the distance between a vortex pair. According to Kirchhoff (1883) the vortex dynamics will be governed by the equations of motion

$$\rho \kappa_i \dot{x}_i = \partial H/\partial y_i, \qquad \rho \kappa_i \dot{y}_i = -\partial H/\partial x_i. \tag{6}$$

Although (5) is correct only for an unbounded fluid, the theory has been extended by Lin (1943) to include certain boundaries in the 2D fluid, and Fetter (1967) has shown how a Hamiltonian formalism also follows for small oscillations of nearly rectilinear vortices in 3D. Our considerations will be restricted to the cases of an isolated vortex and then to a vortex pair in an unbounded film, and we will quantise equations (2) and (3). For an isolated vortex our quantum model predicts that (1) describes the average velocity outside the core, i.e. when $r \gg a$, and that (2) describes an interacting vortex pair whenever $r \gg a$, where a is the length scale determined by quantum fluctuations in the position of a vortex (i.e. by the vortex core as determined by the non-commuting operators $\hat{X}_i$ and $\hat{Y}_i$). However, this limiting classical behaviour is achieved asymptotically very rapidly as a function of r/a, so that our results suggest that it is a good approximation to use the classical model

$$H \simeq \begin{cases} -(\rho\kappa_1\kappa_2/2\pi)\log(r_{12}/a), & r_{12} > a \\ 0, & r_{12} < a \end{cases} \tag{7}$$

for attracting vortices ($\kappa_1 = -\kappa_2$). That is, a 'soft-core' model is predicted in contrast with the hard-core model used in the literature (Kosterlitz and Thouless 1973), but the distinction is not significant for the purpose of calculation. What is important is that quantisation yields a model which is essentially classical outside the vortex core, but at the same time provides a definite model for the vortex core based upon quantum fluctuations. The detailed short-distance predictions of the model must be viewed with caution, but are a conceptual advance beyond the classical theory as applied to thin ^{4}He films, where the vortex core size is undetermined and is introduced in an *ad hoc* fashion.

3. Quantum fluctuations in the position of a vortex, and phase space quantisation

Our considerations are motivated by attempting to use equation (3) to describe the velocity near a quantised vortex by replacing the 2D delta function $\delta(\boldsymbol{r}-\boldsymbol{r}')$ by a self-adjoint operator that represents the vortex density or vorticity. This is required by the fact that $\boldsymbol{r}'$ is to be replaced by an operator, and is complicated by the fact that X' and Y' must be replaced by non-commuting operators $\hat{X}$ and $\hat{Y}$ such that $\hat{q}=\sqrt{\rho\kappa}\,\hat{X}$ and $\hat{p}=\sqrt{\rho\kappa}\,\hat{Y}$ obey the commutation rule $[\hat{q},\hat{p}]=\mathrm{i}\hbar$. Since the vorticity will have quantum fluctuations, so will the velocity, but we can define average values by taking expectation values of the appropriate operators. The first goal is to find the correct operators.

We are required to replace the classical 2D delta function $\delta(\boldsymbol{r}-\boldsymbol{r}')\sim\delta(q-q')\delta(p-p')$ by an operator when $(q',p')\rightarrow$ operators $(\hat{q},\hat{p})$†. If we Fourier-transform the classical delta function,

$$\delta(q-q')\delta(p-p')=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p-\mathrm{i}\xi q'-\mathrm{i}\eta p'},\tag{8}$$

then the most obvious correspondence $\delta(q-q')\delta(p-p')\rightarrow\hat{\Delta}$ is given by simply replacing q' and p' by $\hat{q}$ and $\hat{p}$ in (8),

$$\hat{\Delta}=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}\xi q+\mathrm{i}\eta p}\,\mathrm{e}^{-\mathrm{i}\xi\hat{q}-\mathrm{i}\eta\hat{p}},\tag{9a}$$

which is just the Fourier transform of the unitary operator $\mathrm{e}^{-\mathrm{i}\xi\hat{q}-\mathrm{i}\eta\hat{p}}$. The expectation value of this operator yields Wigner's distribution (Agarwal and Wolf 1970) for the vorticity, and corresponds to Weyl's rule for forming quantum mechanical operators from classical functions defined on phase space (Weyl 1931, Agarwal and Wolf 1970). This method of quantisation is well known, and has been studied extensively in the literature (Agarwal and Wolf 1970, Srinivas and Wolf 1975). Equations (8) and (9*a*) are an example of a case where the method arises rather naturally. However, Wigner's distribution is known to take negative values for certain values of the variables (q,p) (note that the field point in the fluid is given by $\boldsymbol{r}=(\rho\kappa)^{-1/2}(q,p)$). For example, the Wigner distribution is Gaussian for thermal and zero-point motions of an oscillator (Feynman 1972), but becomes negative for large q or p for oscillator states $|l\rangle$, where l is odd (l is the oscillator quantum number). Since the vortex has rotational symmetry, its position fluctuations will be represented by oscillator states, and so we conclude that Wigner's distribution is not a suitable candidate for the vortex density, which must be positive semi-definite (this is necessary if $\kappa>0$; if $\kappa<0$ the vortex density is negative semi-definite).

Now it is well known that there are many phase space quantisation methods, so that the Weyl symmetrisation rule is not unique (Srinivas and Wolf 1975). This non-uniqueness is only one example of the non-uniqueness of quantum mechanical sym-

† Operators are denoted by symbols with 'hats'.

metrisation rules in general. These problems arise because there is a many-to-one correspondence between classical polynomials $p^n q^m$ and self-adjoint operators which are polynomials of order n and m in $\hat{p}$ and $\hat{q}$ (Groenewold 1946, Shewell 1959). In general, there are many self-adjoint operators with the same classical limit as $\hbar \to 0$. Because of our starting point (8), we are interested quite naturally in the predictions of the phase space symmetrisation rules. In particular, how many of these rules predict a vortex density with definite sign? There is only one symmetrisation rule that fills this requirement, and to illustrate this we will first discuss several alternatives.

The non-uniqueness of quantum operators corresponding to (8) can be seen in the following way. Rather than the choice (9*a*), suppose that we had made the replacement

$$e^{-i\xi q'-i\eta p'} \to (e^{-i\xi\hat{q}}\, e^{-i\eta\hat{p}} + e^{-i\eta\hat{p}}\, e^{-i\xi\hat{q}})/2$$

in (8). Then the analogue of (9*a*) would be

$$\hat{\Delta} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi\, d\eta\, \cos\left(\frac{\hbar\xi\eta}{2}\right) e^{i\xi(q-\hat{q})+i\eta(p-\hat{p})}, \tag{9b}$$

which is the rule corresponding to 'standard' symmetrisation (Agarwal and Wolf 1970). It predicts an oscillatory Wigner-like distribution

$$w_0(q, p) = \sqrt{2/\hbar}\, e^{-(q^2+p^2)/2\hbar} \cos(pq/\hbar) \tag{10}$$

for the zero-point motion, where $w_0(q, p) = \langle 0|\hat{\Delta}|0\rangle$, and so we discard it as a model for the vortex density. In general, we must consider the replacement

$$e^{i\xi q'+i\eta p'} \to \sigma(\xi, \eta)\, e^{i\xi\hat{q}+i\eta\hat{p}}, \tag{11a}$$

where $\sigma(\xi, \eta)$ is a sum of phase factors arising from the choice of symmetrisation rule (Srinivas and Wolf 1975). $\sigma = 1$ corresponds to Weyl–Wigner, $\sigma = \cos(\xi\eta\hbar/2)$ to standard ordering of operators, $\sigma = e^{(\xi^2+\eta^2)\hbar/4}$ to normal ordering, where $e^{i\xi q'+i\eta p'} \to e^{-Z^*\hat{a}^+}\, e^{Z\hat{a}}$ with $\hat{a} = (\hat{q} + i\hat{p})(2\hbar)^{-1/2}$ and $Z = \sqrt{\hbar/2}(\eta + i\xi)$. None of these gives rise to a positive semi-definite operator $\hat{\Delta}$, but it is already known that this requirement is fulfilled by the rule for anti-normal ordering (Srinivas and Wolf 1975), according to which

$$e^{i\xi q'+i\eta p'} \to e^{z\hat{a}}\, e^{-z^*\hat{a}^+} = e^{-|z|^2/2+z\hat{a}-z^*\hat{a}^+}, \tag{11b}$$

which corresponds to (11*a*) with

$$\sigma(\xi, \eta) = e^{-|z|^2/2} = e^{-(\xi^2+\eta^2)\hbar/4}. \tag{11c}$$

Planck's constant $\hbar$ is important in determining the vortex core size, but we will first define the operator $\hat{\Delta}$ in terms of dimensionless variables (Q, P) and dimensionless operators $(\hat{Q}, \hat{P})$, where the latter satisfy $[\hat{Q}, \hat{P}] = i$ (the original canonical variables $\hat{q} = \sqrt{\rho\kappa}\, \hat{X}$ and $\hat{p} = \sqrt{\rho\kappa}\, \hat{Y}$ satisfy $[\hat{q}, \hat{p}] = i\hbar$). Using the anti-normal ordering rule yields the vortex density operator

$$\hat{\Delta} = \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} d\xi\, d\eta\, e^{-\hbar(\xi^2+\eta^2)/4}\, e^{i\xi\hat{Q}+i\eta\hat{P}}\, e^{-i\xi Q-i\eta P}, \tag{9c}$$

which can be transformed into the form

$$\hat{\Delta} = (1/\pi^2) \iint d^2z\, e^{z\hat{a}}\, e^{-z^*\hat{a}^+}\, e^{-\alpha z+\alpha^* z^*}, \tag{9d}$$

where $\alpha = 2^{-1/2}(Q+\mathrm{i}P)$ and $\hat{a} = 2^{-1/2}(\hat{Q}+\mathrm{i}\hat{P})$. It follows easily from (9*d*) (Srinivas and Wolf 1975) that

$$\hat{\Delta} = (1/2\pi)|Q, P\rangle\langle Q, P|, \tag{12}$$

where $|Q, P\rangle$ is a coherent state (Klauder and Sudarshan 1968). $2\pi\hat{\Delta}$ is therefore positive semi-definite since it is the coherent state projection operator. For an arbitrary quantum state $|\psi\rangle$, the average vortex density will be given by the rather appealing result

$$w_\psi(Q, P) = \langle\psi|\hat{\Delta}|\psi\rangle = (1/2\pi)|\langle Q, P|\psi\rangle|^2, \tag{13}$$

which is as close to a 'Q, P-probability density' as quantum theory will permit, and is explicitly non-negative. The main question is whether (13) leads to sensible predictions for the vortex density. Classical hydrodynamics, which must be valid asymptotically, demands that the vorticity must have a finite range a, so that equation (1) is valid for $r \gg a$. We will show next that (13) fulfils this requirement for oscillator eigenstates, and that for an arbitrary oscillator state $|l\rangle$, the vortex core size is determined by the zero-point motion.

Evaluating (13) for oscillator eigenstates $|l\rangle$ yields

$$w_l(Q, P) = \langle l|\hat{\Delta}|l\rangle = [(Q^2+P^2)^l/2\pi l!2^l]\,\mathrm{e}^{-(Q^2+P^2)/2}, \tag{14}$$

which is Gaussian for $l = 0$ (zero-point motion). In the latter case the average velocity† is

$$\langle \boldsymbol{v}\rangle_0 = (\kappa\hat{\theta}/2\pi r)(1-\mathrm{e}^{-r^2/a^2}), \tag{15}$$

where

$$a^2 = \langle 0|(\hat{x}^2+\hat{y}^2)|0\rangle = 2\hbar/\rho\kappa \tag{16}$$

gives the vortex core size and is due to zero-point motion. If we assume that the circulation is quantised ($\kappa = h/m$), then, since ρ is the mass per unit area in the 2D fluid, we obtain $a^2 \sim m/\rho$. Therefore the combination of Heisenberg's uncertainty principle and Onsager's circulation quantisation predicts a minimum core size of the order of an interatomic spacing, reflecting the atomic nature of the fluid. Note also that when $r \gg a$ we retrieve equation (1), while if $r \lesssim a$ equation (15) becomes approximately

$$\boldsymbol{v} \approx \kappa\hat{\theta}r/2\pi a^2, \tag{17}$$

which is the classical formula for a vortex core in solid-body rotation. However, if $a \sim 4$ Å, there is no actual rotation of fluid within the core, and the correct way to understand (17) is that it follows from position fluctuations of the vortex†. It also indicates that the kinetic energy density of the fluid will vanish (rather than diverge) as $r \to 0$.

The model of a quantised vortex that we have just described is closely related to Fetter's (1967) model, but restricted to two dimensions. Fetter's considerations were based initially upon a discontinuous distribution of classical vorticity, since that distribution can be interpreted as a model for the self-energy of a singular vortex (Fetter 1967). Our average vorticity is Gaussian for the zero-point motion, and is very well approximated by a discontinuous distribution. However, in the 3D model the parameter a can become much larger than an inter-particle spacing due to thermal

† The details are given in the Appendix.

motions of the line (Fetter 1967), although not in the case of a 2D vortex (see equation (20) below). It is interesting to see what will be the core size as predicted by (16) near the critical point for vortex pair dissociation. According to Nelson and Kosterlitz (1977) the superfluid density ρ in a thin film will undergo a discontinuous jump at the critical point, and the condition for the critical point is that $\rho/T = 3{\cdot}52\times10^{-9}\,\mathrm{g\,cm^{-2}\,K^{-1}}$. If $T\sim1$ K, this yields a core size of the order of 4 Å, not significantly different from the zero-temperature estimate, but this leaves unanswered the question of a discontinuous core size change at the critical point, and also does not consider the possible contributions to the core from other excitations in the fluid. Ambegaokar *et al* (1978) have stated that a core size of the order of an angstrom is consistent with the superfluid density measurements of Bishop and Reppy (1978), but this is not a conclusive test of the prediction.

It is not hard to see that thermal fluctuations in vortex position are not likely to increase the core size significantly. For a vortex in thermal equilibrium we must calculate the thermally averaged vorticity, which is proportional to the Wigner-like distribution

$$w(x, y) = \kappa\ \mathrm{Tr}(\mathrm{e}^{-\beta\hat{H}_s}\hat{\Delta}/Z). \tag{18}$$

$\mathrm{e}^{-\beta\hat{H}_s}/Z$ is the statistical operator, and $\hat{H}_s$ is the self-energy operator for a vortex in a film of finite thickness L. If we adopt Fetter's (1967) model for the self-oscillations, with which we are already in qualitative agreement, the vortex motion is represented by a superposition of quantised oscillations with wavelengths in the range $a<\lambda<L$, and the (discrete) number of such modes will be small, because L is of the order of a. For the purposes of illustrating the effect it is entirely sufficient to consider one mode with $\lambda\sim L\sim a$. The resulting vortex density is Gaussian (Power 1978),

$$w(x, y) = (\kappa/\pi a^2)\,\mathrm{e}^{-(x^2+y^2)/a^2}, \tag{19}$$

and the corresponding vortex core size is given by

$$a^2 = (2\hbar/\rho\kappa)(1-\mathrm{e}^{-\beta\hbar\omega})^{-1}, \tag{20}$$

where $\hbar\omega\sim\rho\kappa^2/2\pi$ whenever $\lambda\sim a$. Since $\beta\hbar\omega>4$ whenver $T<T_c$ (T_c is the 2D vortex dissociation temperature in the ^{4}He film (Nelson and Kosterlitz 1977)), we retrieve our zero-point core size as given by (16) above. In the case of bulk ^{4}He, there are slow hydrodynamic modes with $\lambda\sim L\sim1$ cm (Fetter 1967), and one can see qualitatively from (20) that very large position fluctuations will follow.

We will now consider the effect of uncertainty in vortex position upon the usual statement of the circulation theorem. Since we have a vortex whose location is fluctuating randomly over a small area $\sim a^2$ on the average, it is clear that the traditional statement of circulation quantisation (Feynman 1972) will not hold precisely. However, a reasonable generalisation is still valid. Denoting the average circulation by

$$\Gamma = \oint_C \langle v\rangle\,.\,\mathrm{d}l = \iint w(x, y)\,\mathrm{d}x\,\mathrm{d}y, \tag{21}$$

where $w(x, y)$ is the average vorticity and C is a loop enclosing the average location of the vortex ($\langle X\rangle = \langle Y\rangle = 0$), we obtain for a circle of radius r the circulation

$$\Gamma = \kappa(1-\mathrm{e}^{-r^2/a^2}) \tag{22}$$

whenever $w(X, Y)$ is Gaussian. Classically we would have used $w(X, Y) = \kappa\delta(r)$ to obtain $\Gamma=\kappa$. Equation (22) reflects the fact that the vortex has not been enclosed with certainty within our circle C, and Γ/κ is the probability of finding the vortex inside C. If $r=\infty$, the vortex is certainly enclosed by C and we have $\Gamma(\infty)=\kappa$.

For an arbitrary oscillator state, the vorticity is not Gaussian but is the $2n$th derivative of a Gaussian, as can be seen from (14). One must ask whether such a distribution gives the correct normalisation of vorticity, which is to say that we have one vortex with circulation κ somewhere in an infinite fluid. The answer here is in the affirmative, and is guaranteed by the general theorem (Agarwal and Wolf 1970)

$$\iint_{-\infty}^{\infty} \mathrm{d}Q\,\mathrm{d}P\,\hat{\Delta}=\hat{I}, \tag{23}$$

where $\hat{I}$ is the identity operator. Equation (23) is just the 'over-completeness' relation for coherent states (Klauder and Sudarshan 1968) written in terms of the dimensionless variables (Q, P).

We will now turn to the task of working out the predictions of our model for a pair of interacting vortices. To that end we will need the general rules for phase space quantisation, which can be stated as follows. To each classical phase space function $G(q,p)$ there corresponds a self-adjoint operator given by the following rule. First obtain the Fourier transform of $G(q,p)$,

$$\gamma(\xi,\eta)=\iint \mathrm{d}q\,\mathrm{d}p\,\mathrm{e}^{-\mathrm{i}\xi q-\mathrm{i}\eta p}G(q,p). \tag{24a}$$

In the 'inverse transform' the replacement (11b) gives the operator $\hat{G}$ corresponding to G:

$$\hat{G}(\hat{q},\hat{p})=\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty}\mathrm{d}\xi\,\mathrm{d}\eta\,\gamma(\xi,\eta)\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}-(\xi^2+\eta^2)\hbar/4}. \tag{24b}$$

4. Quantisation of vortices with equal circulation

Consider a pair of vortices with equal circulation ($\kappa_1=\kappa_2=\kappa$) in an unbounded fluid. We will assume $\kappa>0$ for simplicity. If we transform from (q_1,q_2,p_1,p_2) to 'relative' and 'centre-of-mass' variables $(q,p;Q_c,P_c)$, where $q=2^{-1/2}(q_1-q_2)$, $p=2^{-1/2}(p_1-p_2)$, $Q_c=2^{-1/2}(q_1+q_2)$ and $P_c=2^{-1/2}(p_1+p_2)$, then $\{q,p\}=\{Q_c,P_c\}=1$, while all other Poisson brackets vanish. The classical Hamiltonian H depends only upon the relative coordinates $q=\sqrt{\rho\kappa}(X_1-X_2)$ and $p=\sqrt{\rho\kappa}(Y_1-Y_2)$:

$$H=-(\rho\kappa^2/4\pi)\log(q^2+p^2) \tag{25}$$

to within a constant. Because $\{q,p\}=1$, we have the symmetry of an oscillator problem. In spite of the repulsive nature of the interaction, two such vortices rotate about one another with constant separation (McCormack and Crane 1973), where the vortex separation r is proportional to $(q^2+p^2)^{1/2}$, and the quantisation of separation follows immediately (Mittag and Stephen 1968). We now turn to the task of working out the spectrum of (25) as predicted by phase space quantisation, using anti-normal ordering as the symmetrisation rule.

Since (25) depends only upon the canonically conjugate variables (q,p), but not upon (Q_c,P_c), the operator $\hat{H}$ corresponding to (25) will be given directly by (24b) if we replace $\gamma(\xi,\eta)$ by the Fourier transform of (25). Now $\log r$ in two dimensions does not have a Fourier transform in the strict sense, but we can define it formally by Laplace's equation

$$\nabla^2 H(q, p) = -\rho\kappa^2 \delta(q)\delta(p), \tag{26}$$

where $\nabla^2 = \partial^2/\partial q^2 + \partial^2/\partial p^2$, yielding

$$\gamma(\xi, \eta) = \rho\kappa^2(\xi^2+\eta^2)^{-1} \tag{27}$$

and

$$\hat{H} = \frac{\rho\kappa^2}{(2\pi)^2}\iint_{-\infty}^{\infty} \frac{\mathrm{d}\xi\,\mathrm{d}\eta}{\xi^2+\eta^2}\,\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}-(\xi^2+\eta^2)\hbar/4}, \tag{28}$$

where $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$. Equation (28) is logarithmically divergent when $\xi = \eta = 0$ (i.e. at infinite wavelength or infinite separation), but this divergence is a c-number; it can be subtracted from the operator part of (28), which is well-defined (see (31) below). Furthermore, this logarithmic divergence corresponds to the infinite kinetic energy of a classical vortex in an unbounded fluid (because it occurs for infinite wavelength), and occurs for the same reason that (27) cannot be interpreted literally as the Fourier transform of (25) (i.e. without a convergence factor). We note that, while (27) is not the Fourier transform of (25) in the strict sense, it can be defined as such in the theory of generalised functions (Shilov 1968), and according to Weyl (1931) we should not always expect to interpret the Fourier transform literally. By ignoring this c-number divergence we are led to a model which is superior to the classical point vortex model that we used as a starting point: namely, there is no divergence of (28) when $\xi = \eta = \infty$ corresponding to the classical logarithmic divergence when $q = p = 0$ ($\xi, \eta = \infty$ corresponds to vanishing wavelength or short distances). The latter divergence is eliminated from (28) by the zero-point motion that follows from the rule $[\hat{q}, \hat{p}] = \mathrm{i}\hbar$. To see this we must separate the operator part of (28), which is well-defined, from the logarithmically divergent c-number. To this end we transform to the usual ladder operators $\hat{a} = (2\hbar)^{-1/2}(\hat{q}+\mathrm{i}\hat{p})$ and $\hat{a}^+ = (2\hbar)^{-1/2}(\hat{q}-\mathrm{i}\hat{p})$, where $[\hat{a}, \hat{a}^+] = 1$, and to complex variables Z and Z^* defined by $Z = (\hbar/2)^{1/2}(\eta+\mathrm{i}\xi)$. If we use

$$\mathrm{e}^{\mathrm{i}\xi\hat{q}+\mathrm{i}\eta\hat{p}} = \mathrm{e}^{-z^*\hat{a}^+ + z\hat{a}} = \mathrm{e}^{-z^*\hat{a}^+}\,\mathrm{e}^{z\hat{a}-|z|^2/2}, \tag{29}$$

equation (28) becomes

$$\hat{H} = \frac{\rho\kappa^2}{(2\pi)^2}\iint_{-\infty}^{\infty} \frac{\mathrm{d}^2 Z}{|Z|^2}\,\mathrm{e}^{-|Z|^2}\,\mathrm{e}^{-Z^*a^+}\,\mathrm{e}^{Za}. \tag{30}$$

Setting $Z = r\,\mathrm{e}^{\mathrm{i}\phi}$ and expanding the operators $\mathrm{e}^{-Z^*\hat{a}^+}$ and $\mathrm{e}^{Z\hat{a}}$ in power series yields (aside from the infinite c-number)

$$\hat{H} = -\frac{\rho\kappa^2}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nn!}\hat{N}(\hat{N}-1)(\hat{N}-2)\ldots[\hat{N}-(n-1)], \tag{31}$$

where $\hat{N} = \hat{a}^+\hat{a}$ is the oscillator number operator and we have used the relation

$$a^{+n}a^n = \hat{N}(\hat{N}-1)\ldots[\hat{N}-(n-1)], \tag{32}$$

which can be proven by induction and has been quoted in the literature (Klauder and Sudarshan 1968). Since $\hat{N}|l\rangle = l|l\rangle$, where $|l\rangle$ is an oscillator eigenstate and

$l = 0, 1, 2, \ldots$, we obtain for the eigenvalue $E_l = \langle l|H|l\rangle$ of H the result

$$E_l = -\frac{\rho\kappa^2}{4\pi}\sum_{n=1}^{l}\frac{(-1)^{n+1}}{n}\binom{l}{n}, \tag{33}$$

showing that E_l is 'quadratic' in the distance (rather than logarithmic) at short distances ($r^2 \sim \langle l|\hat{q}^2 + \hat{p}^2|l\rangle = l + \frac{1}{2}$ is the distance squared between vortices, to within a constant), and we have a finite distance of closest approach when $l = 0$ due to zero-point motion (Mittag and Stephen 1968).

We can obtain a result that is more useful than (33) by integrating the generating function

$$\frac{1-(1-x)^l}{x} = \sum_1^l \binom{l}{n}(-1)^{n+1}x^{n-1} \tag{34}$$

from $x = 0$ to $x = 1$. This yields

$$E_l = -(\rho\kappa^2/4\pi)[\gamma + \psi(l+1)], \tag{35a}$$

where γ is Euler's constant and ψ is Euler's digamma function (Magnus *et al* 1966). Using (35a), it is easy to obtain the asymptotic properties of the spectrum and also the energy level spacing. Since $\psi(Z) \sim \log Z + O(1/Z)$ as $|Z| \to \infty$ (Magnus *et al* 1966), we retrieve the classical logarithmic behaviour

$$E_l \sim -(\rho\kappa^2/4\pi)\psi(l+1) \sim -(\rho\kappa^2/4\pi)\log l \tag{36}$$

in the limit of large distances $r \sim l^{1/2} \to \infty$. Since $\psi(Z+1) - \psi(Z) = 1/Z$, the energy level spacing is given by

$$E_{l+1} - E_l = -(\rho\kappa^2/4\pi)/(l+1) \tag{37a}$$

whenever $l \geqslant 1$, showing both the monotonic behaviour of E_l with distance as well as the asymptotically logarithmic behaviour at large distances.

The 'quadratic' behaviour of E_l for small l has its physical basis in the nature of the vortex core, as given by (17): since the velocity of a vortex is linear in r whenever $r \leqslant a$, it follows that the kinetic energy will be quadratic in r.

Had we used Weyl symmetrisation rather than anti-normal ordering as our starting point, the Gaussian factor $e^{-|Z|^2}$ in (30) would be replaced by $e^{-|Z|^2/2}$, and the consequence would be an additional factor of 2^n in (31) and (33). The analogue of (35a) is obtained by integrating a generating function analogous to (34), and yields

$$E_l = -\frac{\rho\kappa^2}{4\pi}\left\{\psi(l+1) + \gamma + \log 2 + \frac{(-1)^{l+1}}{2}\left[\psi\left(\frac{l+2}{2}\right) - \psi\left(\frac{l+1}{2}\right)\right]\right\}, \tag{35b}$$

with the energy level spacing (for $l \geqslant 1$)

$$E_{l+1} - E_l = -\frac{\rho\kappa^2}{4\pi}\left(\frac{1+(-1)^l}{l+1}\right). \tag{37b}$$

Therefore this model produces unphysical results in two respects: the average vorticity is not positive semi-definite, and the energy spectrum (35b) contains a strange degeneracy ($E_{2l} = E_{2l-1}$, if $l = 1, 2, 3, \ldots$) which contradicts our expectation that a larger separation between repelling vortices ought to correspond to a lower energy. On the other hand, the model given by anti-normal ordering is not only in agreement with classical expectations where it ought to be, but predicts an origin for the vortex core based upon the quantum fluctuations in position of a 'point vortex'.

5. Quantisation of vortices with opposite circulation

We turn now to the consideration of a pair of attracting vortices with circulations $\kappa_1 = -\kappa_2 = \kappa > 0$. The classical Hamiltonian is given by

$$H = (\rho\kappa^2/2\pi)\log r_{12}, \tag{2b}$$

where q_1, q_2 and p_1 are the same as before, but now we have $p_2 = -\sqrt{\rho\kappa}\, y_2$. This seemingly trivial change of sign reflects a great difference in the physics of two vortices: classically two vortices with opposite circulation translate with constant velocity perpendicular to their (constant) separation (McCormack and Crane 1973), and quantisation will lead us to a plane wave problem rather than to an oscillator problem. As before, we transform to variables (q, p, Q_c, P_c) defined by $q = 2^{-1/2}(q_1 - q_2)$, $p = 2^{-1/2}(p_1 - p_2)$, $Q_c = 2^{-1/2}(q_1 + q_2)$ and $P_c = 2^{-1/2}(p_1 + p_2)$, and we have $\{q, p\} = \{Q_c, P_c\} = 1$, while all other Poisson brackets vanish. However, whereas for the case $\kappa_1 = \kappa_2$ we had $q = 2^{-1}(\rho\kappa)^{1/2}(x_1 - x_2)$, $p = 2^{-1}(\rho\kappa)^{1/2}(y_1 - y_2)$ and a Hamiltonian given as a function of $q^2 + p^2$, where $\{q, p\} = 1$, we now have $q = (2^{-1}\rho\kappa)^{1/2}(x_1 - x_2)$, but $P_c = (2^{-1}\rho\kappa)^{1/2}(y_1 - y_2)$, where $\{q, P_c\} = 0$, and the classical Hamiltonian is given (to within a constant) by

$$H(q, P_c) = (\rho\kappa^2/4\pi)\log(q^2 + P_c^2). \tag{38}$$

To obtain the quantum mechanical Hamiltonian $\hat{H}$, we must begin with the generalisation of (24*b*) to four canonical variables, and require also the Fourier transform of (38) with respect to all four variables (q, p, Q_c, P_c). In particular, there will be a factor $\sigma(\xi_i, \eta_i)$ with each of two unitary operators $e^{i\xi_i\hat{q}_i + i\eta_i\hat{p}_i}$, but since $\gamma(\xi_1, \eta_1, \xi_2, \eta_2)$ contains two delta functions (Q_c and p are absent from (38)), the final result is a two-dimensional integral, but with a different Gaussian than in the previous case (compare with equation (28)):

$$\hat{H} = -\frac{\rho\kappa^2}{4\pi^2}\iint_{-\infty}^{\infty} \frac{d\xi\, d\eta\, e^{i\xi\hat{q} + i\eta\hat{P}_c - (\xi^2+\eta^2)\hbar/2}}{\xi^2 + \eta^2}, \tag{39}$$

where $\hat{q}$ and $\hat{P}_c$ are *commuting* translation operators and (as before) we must subtract a logarithmically divergent *c*-number from the operator part of (39). We now make a transformation to non-self-adjoint operators

$$\hat{a} = (2\hbar)^{-1/2}(\hat{q} + i\hat{P}_c), \qquad \hat{a}^+ = (2\hbar)^{-1/2}(\hat{q} - i\hat{P}_c), \tag{40}$$

but since $[\hat{a}, \hat{a}^+] = 0$, $\hat{a}$ and $\hat{a}^+$ are not ladder operators. The remaining operations are the same as in § 4, and yield

$$\hat{H} = -\frac{\rho\kappa^2}{4\pi}\sum_1^{\infty}(-1)^{n+1}(nn!)^{-1}(\hat{a}^+\hat{a})^n. \tag{41}$$

The Hamiltonian is diagonal with respect to translation eigenstates $|q, P_c\rangle$, where the eigenvalues (q, P_c) of $(\hat{q}, \hat{P}_c)$ vary continuously from $-\infty$ to ∞, yielding the distance $r = (\rho\kappa)^{-1/2}(q^2 + P_c^2)^{1/2}$ as a good quantum number which varies continuously from

$r=0$ to ∞. Since the energy is given by $E(r)=\langle q, P_c|\hat{H}|q, P_c\rangle$ according to

$$E(r)=\frac{\rho\kappa^2}{4\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{nn!}\left(\frac{\rho\kappa r^2}{2\hbar}\right)^n, \tag{42a}$$

we can use the expansion for the exponential integral $E_1(r^2/a^2)$ (Magnus *et al* 1966) to show (with a given by (16)) that

$$E(r)=(\rho\kappa^2/4\pi)[\gamma+\log(r/a)^2+E_1(r^2/a^2)]. \tag{42b}$$

Equations (42a) and (42b) suggest that, for the purpose of classical equilibrium statistics (Kosterlitz and Thouless 1973) and kinetic theory (Ambegaokar *et al* 1978), it is a good approximation to replace $E(r)$ by the soft-core model (7) above, where we count the vortices as having annihilated when $r<a$ (note that $E(0)=0$ corresponds physically to the absence of kinetic energy in the fluid).

We can also obtain the quantum analogue of the classical theorem that two vortices with opposite circulation translate with constant velocity perpendicular to their (constant) separation. To see this it is necessary only to note that $\hat{x}_1-\hat{x}_2\alpha\hat{q}=0$ and $\hat{y}_1-\hat{y}_2\alpha\hat{P}_c=0$, since $\hat{q}$ and $\hat{P}_c$ are conserved ($\hat{x}_i$ and $\hat{y}_i$ are now assumed to be given in the Heisenberg representation). It follows directly from the Heisenberg equations of motion for $\dot{\hat{x}}_i$ and $\dot{\hat{y}}_i$ that $\dot{\hat{x}}_i$ and $\dot{\hat{y}}_i$ are functions only of the operators $\hat{q}$ and $\hat{P}_c$, and therefore are also constants of the motion (i.e. the velocity of the vortex pair is a constant of the motion with a continuous spectrum). Physically, these results are easy to understand: since we have an infinite fluid, we have continuously varying velocities due to translational invariance, and this means that both r and $E(r)$ must vary continuously as well. The case of a bounded fluid is certainly of interest for actual ^{4}He films, and in this case it is possible that the spectrum might be discrete, since the plane wave states will be required to obey periodic or other boundary conditions at the edges of the film.

6. Comparison with predictions from the spectral theorem

We now wish to compare the results of the last three sections with predictions based upon a more familiar line of reasoning; namely, the quantisation of vortices via the spectral theorem.

We begin with the case $\kappa_1=\kappa_2=\kappa$. Since $\hat{q}^2+\hat{p}^2=\hat{N}+\frac{1}{2}$, where $\hat{N}$ is the oscillator number operator, if we assume that arbitrary functions of $\hat{N}$ can be defined by

$$f(\hat{N})=\sum_{n=0}^{\infty}f(n)|n\rangle\langle n|, \tag{43}$$

where f is the classical function of q^2+p^2, then it follows that the Hamiltonian for a repelling vortex pair is given by

$$\hat{H}=-\frac{\rho\kappa^2}{4\pi}\sum_{n=0}^{\infty}\log(n+\tfrac{1}{2})|n\rangle\langle n|, \tag{44}$$

with eigenvalues

$$E_n=-(\rho\kappa^2/4\pi)\log(n+\tfrac{1}{2}). \tag{45}$$

This forms the prediction of what we have called the Mittag–Stephen model (Mittag and

Stephen 1968). In contrast, the phase space model predicts that

$$E_n = -(\rho\kappa^2/4\pi)[\psi(n+1)+\gamma]. \tag{35a'}$$

Both (35a') and (45) are monotonic and asymptotically logarithmic, so that in their gross features they do not disagree. The disagreement comes primarily at small distances (small n), where neither theory can be trusted beyond question, since both are based upon quantising hydrodynamics. It is interesting to note that (45) would follow from von Neumann's rules for a classical–quantal correspondence (Groenewold 1946, Shewell 1959), since those rules include the two requirements $A+B \to \hat{A}+\hat{B}$ and $f(A) \to f(\hat{A})$, where A and B are classical quantities, while $\hat{A}$ and $\hat{B}$ are self-adjoint operators. We will now use this rule to extend the model, and will see that the results become less impressive.

First, let us ask for the operator representing the vorticity. The following reasoning leads to Wigner's distribution: if $(q, p) \to (\hat{q}, \hat{p})$, then $\xi q + \eta p \to \xi\hat{q} + \eta\hat{p}$ and $f(\xi q + \eta p) \to f(\xi\hat{q} + \eta\hat{p})$, which requires that $e^{i(\xi q+\eta p)} \to e^{i(\xi\hat{q}+\eta\hat{p})}$, yielding Wigner's distribution for the average vorticity. We rejected this distribution previously because it can become negative.

Second, if we consider two vortices with $\kappa_1 = -\kappa_2$, then according to (43) the energy eigenvalues are given by the classical result

$$E = (\rho\kappa^2/4\pi)\log(q^2 + P_c^2), \tag{46}$$

where q and P_c vary continuously in the range $-\infty \leqslant q, P_c \leqslant \infty$, and we obtain the short-distance divergence of energy characteristic of classical vortices with no finite core. This result is somewhat analogous to the motion of a free particle of finite mass in free space: in both cases the prediction of a classical spectrum follows from translational invariance.

We have shown that only in the case where the quantum mechanical operators follow from anti-normal ordering is the resulting model free *simultaneously* from (i) the physically unrealistic small-distance divergence in energy and velocity characteristic of a classical point vortex, and (ii) the physically unrealistic occurrence of negative densities. The result (12) suggests that it may be possible to develop the theory entirely from the coherent state formalism without reference to phase space quantisation, but we have not pursued this.

7. Summary

We have shown that it is possible to start from the classical canonical dynamics of 2D point vortices and use a particular symmetrisation rule to construct a quantum model of point vortices which is free from the classical short-distance divergences in energy and velocity. This was accomplished by constructing an appropriate operator to represent the vorticity, or vortex density, and led to a theory where the quantised point vortex has a 'core' because of the uncertainty principle. We have shown how the Onsager–Feynman circulation quantisation theorem fits into our picture via a simple generalisation, and how the combination of Heisenberg's uncertainty principle and Onsager's circulation quantisation predicts a minimum core size equal to an inter-particle spacing, reflecting (even in quantised hydrodynamics) the true atomic nature of the fluid.

We have calculated the energy spectrum for a pair of vortices and have shown (in qualitative agreement with Mittag and Stephen) that it is discrete for vortices of equal circulation. For vortices of opposite circulation we have obtained the entirely new prediction of a continuous spectrum. In the former case two vortices have a distance of closest approach dictated by the zero-point motion, whereas in the latter case the vortices may come together and annihilate (vortices with $r \leqslant a$ may be counted as having annihilated) with vanishing fluid kinetic energy.

For the isolated vortex we showed how the zero-point motion of a vortex leads to a Gaussian distribution of vorticity, a finite vortex core size, and an average velocity that vanishes (rather than diverging) near the average position of the vortex. This result was used to explain the absence of a logarithmic divergence at short distances in the energy of an interacting vortex pair.

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Appendix

Consider a point vortex located at $\boldsymbol{r}'$:

$$\nabla^2 \psi = -\kappa \delta(\boldsymbol{r} - \boldsymbol{r}'), \tag{A1}$$

where ψ is the stream function. If $\boldsymbol{r}'$ is subject to a Gaussian noise source, then the average stream function is given by solving

$$\nabla^2 \langle \psi \rangle = -(\kappa / \pi a^2)\, \mathrm{e}^{-r^2/a^2} = -w(\boldsymbol{r}). \tag{A2}$$

The solution can be written in the form

$$\langle \psi(\boldsymbol{r}) \rangle = -(1/2\pi) \int G(\boldsymbol{r}, \boldsymbol{r}') w(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}', \tag{A3}$$

where $G(\boldsymbol{r}, \boldsymbol{r}') = -\log|\boldsymbol{r} - \boldsymbol{r}'|$, and we will evaluate (A3) by using the expansion

$$\log|\boldsymbol{r} - \boldsymbol{r}'| = \log \frac{1}{r_>} - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{r_<}{r_>} \right)^m \cos m(\phi - \phi'). \tag{A4}$$

It follows after a few manipulations that

$$\langle \psi(\boldsymbol{r}) \rangle = -\frac{\kappa}{2\pi} \left(\log r + \int_r^{\infty} \frac{\mathrm{e}^{-r^2/a^2}}{r}\, \mathrm{d}r \right), \tag{A5}$$

and since $v_\theta = -\partial\psi/\partial r$, we obtain the result

$$\langle v_\theta \rangle = (\kappa / 2\pi r)(1 - \mathrm{e}^{-r^2/a^2}), \tag{A6}$$

which was quoted in the text.

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